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# Note

# Wall-Compatibility Condition for the Solution of the Navier–Stokes Equations

## 1. Introduction

Discrete numerical methods for the solution of the Navier–Stokes equations usually need not only the classic solid-wall boundary conditions but also a value for the density at the wall. The use of a compatibility condition found from the continuity equation is proposed to find this value. An application to Couette flow is made.

Consider the formulation of a numerical solution of the Navier–Stokes equations for two-dimensional compressible laminar or Reynolds-averaged turbulent flow (for convenience Cartesian coordinates are used in this discussion). The momentum equations are of second order in $x$ and $y$ for both the velocity components $u$ and $v$. The same is true for the energy equation in either formulation, for instance, for the static temperature $T$.

The consequences for the boundary conditions are that for each of the dependent variables $u, v, T$ two boundary values have to be prescribed. If the flow past a solid body is being considered, boundary values have to be prescribed on the body surface, and in most cases, at an infinite distance from the surface. The continuity equation serves for the determination of the density $\rho$. Because this equation is of first order in $x$ and $y$, only one boundary value can be prescribed for it in each direction, and that only away from the body surface.

A "Gedanken"-experiment gives the same result. At the body surface the nonslip conditions for both $u$ and $v$ are usually valid; for the temperature a certain value of $T$ can be enforced by heating or cooling. Either a zero (insulated surface) or a nonzero heat flux can be prescribed giving a boundary value of the form $\partial T/\partial y$. In general, however, no way exists physically to enforce a value of the density or the pressure, nor gradients of either, at the surface. The values of these quantities at a given point on the body surface, of course, depend on the body configuration, the free-stream condition, and the thermal wall conditions. They can be changed only by changing the free-stream conditions of $\rho$ or $p$, provided body configuration and flow situation remain fixed.

The foregoing discussion was somewhat simplified in order to clarify the physical situation and its consequences with regard to the necessary boundary conditions. In particular, the characteristic properties of the equations are not governed completely by the fact that second-order derivatives are present. For example, the omission of the



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terms $\partial^2/\partial x^2$ does not necessarily render the system of (steady state) equations parabolic, because the pressure field in most cases determines the characteristic properties of the flow (see, i.e., [1]).

## 2. Compatibility Condition for the Density at the Wall

In numerical solutions for the Navier–Stokes equations boundary values for the density $\rho$ or the pressure $p$ are often introduced at the wall (see, i.e., [2, 3]). This is done for reasons connected with the formulation of the solution algorithm.

A popular choice is a zero pressure gradient normal to the wall directly at the wall, similar to boundary layer flow. As discussed in the preceeding section no boundary condition for $\rho$ or $p$ ought to be prescribed at the wall for physical reasons. A zero pressure gradient normal to the wall, however, is not a bad approximation as long as the Reynolds number is high, so that a boundary layer exists, and as long as this boundary layer is neither separated nor leaving the surface (as, e.g., at the trailing edge of an airfoil or of a wing). In the latter cases an interaction takes place, which is rather strong if the flow is separating, and rather small if the flow smoothly leaves the trailing edge. In both cases, however, the pressure gradient at the wall deviates from zero.

In both cases, too, the interaction strongly influences the drag of the body, because it gives rise to the pressure drag. This part of the drag, which is to be distinguished from the pure friction drag, is made up by the difference of the two large pressure forces at the front part and the rear part of the configuration. Therefore the pressure field has to be computed to high accuracy if the drag is to be computed correctly. From these onsiderations it follows that an accurate handling of the computation problem at the surface is desirable. If the need for a value of $\rho$ or $p$ at the surface exists, only a compatibility condition can serve it without constraining the solution.

Such a condition can be found from the momentum equation in the $y$-direction at the wall (again for convenience Cartesian coordinates are used; for general coordinates see [4]):

$$\left.\frac{\partial p}{\partial y}\right|_{y=0} = -\left.\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right)\right|_{y=0}. \tag{1}$$

The use of this compatibility condition (as, e.g., in [5]) is hampered by the fact that second-order derivatives (mixed and nonmixed) have to be computed numerically with one-sided difference formulae, which leads to large inaccuracies. Incidentally, relation (1) for large Reynolds numbers leads to the zero pressure gradient approximation discussed above, because from boundary–layer considerations it can be seen

$$\left.\frac{\partial p}{\partial y}\right|_{\text{B.L.}} = O\left(\frac{1}{\text{Re}}\right) \approx 0. \tag{2}$$

This, however, is not a direct consequence of the so-called thin-layer approximation of the Navier–Stokes equations [4].

A more natural compatibility condition than Eq. (1) can be found from the continuity equation, if, as is usually done, the problem is treated as unsteady with the steady result found asymptotically for $t \to \infty$. This compatibility condition in form of the continuity equation at the wall reads

$$\left.\frac{\partial \rho}{\partial t}\right|_{y=0} = -\rho \left.\frac{\partial v}{\partial y}\right|_{y=0}, \tag{3}$$

or in terms of the pressure for a perfect gas (note that for prescribed heat flux $\partial T/\partial t \neq 0$ in the transient phase),

$$\left.\frac{\partial p}{\partial t}\right|_{y=0} = p \left.\left(\frac{1}{T}\frac{\partial T}{\partial t} - \frac{\partial v}{\partial y}\right)\right|_{y=0}. \tag{4}$$

By using this compatibility condition only first-order derivatives have to be computed. For $t \to \infty$—steady state—each term reduces to the familiar steady state condition,

$$\frac{\partial \rho}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial T}{\partial t} \to 0: \qquad \left.\frac{\partial v}{\partial y}\right|_{y=0} \to 0. \tag{5}$$

## 3. Application

The above compatibility condition was used in a computation of the plane Couette flow. The method [5] was employed with the changes necessary to implement this condition. Of course, the compatibility condition (3) must be employed at both the fixed ($h/y = 0$) and the moving ($y/h = 1$) wall. The reason for this is that, in this particular case, the mass content of the slit is fixed once and for all with the initial density profile. In contrast to an outer-flow case, where the free-stream value for $\rho$ has to be specified, not even one boundary value of $\rho$ can be specified here.

The computation was made for a slit width of $h = 0.3048 \cdot 10^{-5}$ m, a Mach number of the moving wall $M_1 = 3$, a temperature of the moving wall $T_1 = 293.16$ K, a temperature of the fixed wall $T_0 = 0.5 \cdot T_1$, a reference pressure $p_{\text{ref}} = 101600$ Pa, and a reference density of $\rho_{\text{ref}} = 1.207$ kg/m$^3$. The transport properties were assumed to be constant with $\mu = \mu_1$, $k = k_1$. The results ($\Delta t = 0.196$, steady state after 800 time steps, $\Delta y = 0.05$ h) are given in Fig. 1. Steady state was considered to be reached when the largest root mean square norm of the dependent variables dropped below $10^{-5}$. The exact solution of [6] was recovered with a maximum relative error of less than 0.3 percent for the velocity and less than 0.4 percent for the reduced temperature [7]. The initial mass content with $\rho = \rho_{\text{ref}} = \text{const.}$ in the slit is fully conserved.

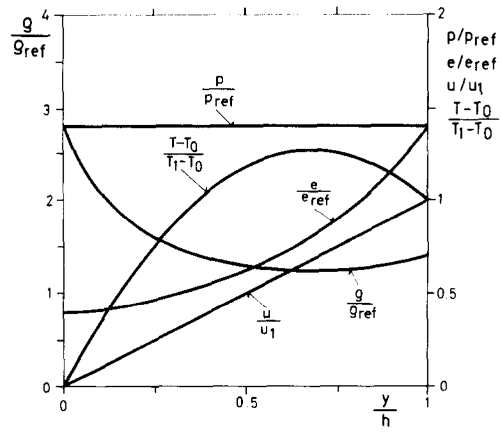


FIG. 1. Results of Couette-flow computations, $M_1 = 3$.

During the transient phase the pressure gradient $\partial p/\partial y$ and hence the velocity component $v$, of course, are nonzero.

## 4. Conclusions

On solid walls no boundary values can be prescribed, in general, for the density $\rho$ or the pressure $p$. If for a numerical algorithm values of $\rho$ or $p$ are needed at the wall, they can be computed by means of a compatibility condition found from the continuity equation. This holds, however, only for unsteady transient and unsteady problems. This compatibility condition has the advantage over the compatibility condition arising from the normal momentum equation in that it contains only first-order derivatives.

The application to the Couette problem has shown that high Mach-number cases with and without insulated walls can be computed [7]. For both walls insulated, of course, no steady state solution exists. Other applications are presently in progress.

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